# OPTIMAL CONTROL OF SYSTEMS WITH RANDOM PROPERTIES

## (OPTIMAL'NOE REGULIROVANIE SISTEM SO SLUCHAINYMI SVOISTVAMI)

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**Systems in which the object of control undergoes random variations are considered. A control law is determined from the conditions of a minimum integral criterion of quality on a finite interval of time. The existence of a solution is discussed. In this work the results obtained in [1,2] are developed for finite intervals of time; in the exposition the terminology and the symbols introduced in [2] are used.**

**1.** Let the transient response of the system be described by the equation

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n, \eta, \xi) \qquad (i = 1, \ldots, n), \qquad \xi = \xi(t, x_1, \ldots, x_n, \eta) \quad (1.1)$$

Here $x = \{x_i(t)\}$ is the error vector of the controlled quantity, $\xi$ is a scalar (the control quantity), $\eta(t)$ is a parameter characterizing the random variations in time of the controlled object. The functions $f_i$ and $\xi$ are continuous and satisfy the conditions of Lipschitz for all the arguments.

The absence of distortions and lags in the loops of the system is assumed, i.e. the controlling quantity $\xi$ at each moment of time is formed on the basis of exact information (the knowledge of $x(t)$ and $\eta(t)$) about the operation of the control.

We shall consider, that the process $\eta(t)$ is Markovian and is either continuous [3, p.307] or purely discontinuous, impulsive [4, p.234]. We shall describe $\eta(t)$ from the given distribution function

$$F(t, \alpha;\ \tau, \beta) = P\{\eta(\tau) \leqslant \beta \setminus \eta(t) = \alpha\} \qquad (\tau \geqslant t)$$

Here $P$ is the conditional probability. We assume that the transfer probability $\eta(t) = \alpha \to \eta(t) \leqslant \beta$ may be broken into

$$P\{\eta(\tau) = \alpha \setminus \eta(t) = \alpha\} = 1 - q(t, \alpha)\Delta t + o(\Delta t)$$
$$P\{\eta(\tau) \leqslant \beta, \eta(\tau) \neq \alpha \setminus \eta(t) = \alpha\} = q(t, \alpha, \beta)\Delta t + o(\Delta t) \tag{1.2}$$

Here $q(t, \alpha)$, $q(t, \alpha, \beta)$ are known functions; $o(\Delta t)$ is a small quantity of order higher than $\Delta t = \tau - t$. Then the realizations $\eta^p(t)$ will be in steps and the process $\eta(t)$ itself is purely discontinuous.

We shall consider the discontinuous process $\eta(t)$ according to the description given in [3], whereupon the existence of the following limits is assumed

$$A_1(t, \alpha) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} [\eta(t + \Delta t) - \alpha]\, d_\eta F(t, \alpha;\ t + \Delta t, \eta) \tag{1.3}$$

$$B_1(t, \alpha) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} [\eta(t + \Delta t) - \alpha]^2 d_\eta F(t, \alpha; t + \Delta t, \eta) \tag{1.4}$$

Let us assume, that all acceptable values of $\eta$ lie* in the interval $\eta_1 < \eta < \eta_2$ (whereupon it is possible to have $\eta_2 = -\eta_1 = \infty$).

We shall now set some functions $\omega(t, x, \xi)$ and $r[x(t)]$ positive definite with respect to $x_1, \ldots, x_n$ and which determine the criterion of quality of the transient response

$$J(x_0, \eta_0; T) = M\left\{\int_0^T \omega(t, x, \xi)\, dt + r[x(T)] \setminus x_0, \eta_0\right\} \tag{1.5}$$

where $M$ is the symbol of the conditional mathematical expectation for the initial conditions $t_0 = 0$, $x_0$, $\eta_0$.

The problem consists in finding a controlling quantity $\xi = \xi^0$ such, that for the system (1.1) the value of the functional (1.5) is minimal for each $x_0$ and $\eta_0 \in (\eta_1, \eta_2)$ when $\xi = \xi^0$.

In the solution of the system (1.1) we shall assume that the vector function $\{t, x(t), \eta(t), \xi(t, x(t), \eta(t))\}$ which can be strongly definite, is a solution of the integral equations, corresponding to the system (1.1) [4, p.248]. It is possible to represent the solution $x(t)$

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* **For a discontinuous process $\eta(t)$, it is possible to assume that the variations of $\eta$ occur on the interval $\eta_1 \leqslant \eta \leqslant \eta_2$.**

as consisting of the realizations of $x^P(t)$, resulting from the realizations of $\eta^P(t)$ and satisfying (1.1).

The general approach to the solution of problems of such a type, based on the method of functions of Liapunov with the use of the notion of dynamic programming [5], is stated in detail in [1, 2]. The singularities, appearing in the considered case, are described below.

2. Let us assume that a function $v(t, x, \eta)$ satisfying the following requirements is found

a) The function $v(t, x, \eta)$ is continuous with respect to all arguments.

b) The derivative $dM\{v\}/dt$ of the mathematical expectation $v(t,x,\eta)$ satisfies the conditions [2]

$$\left(\frac{dM\{v\}}{dt}\right)_{\xi^\circ} + \omega(t, x, \xi^\circ) = 0 \tag{2.1}$$

$$\left(\frac{dM\{v\}}{dt}\right)_{\xi^\circ} + \omega(t, x, \xi^\circ) = \min_\xi \left[\frac{dM\{v\}}{dt} + \omega\right] \tag{2.2}$$

c) For $t = T$, $\eta \in (\eta_1, \eta_2)$ and all values of $x$, the equality

$$v(T, x, \eta) = r[x] \tag{2.3}$$

is fulfilled.

The functions $v^\circ$ and $\xi^\circ$, satisfying these conditions, are solutions of the given problem, i.e. $\xi^\circ$ represents the optimal controlling quantity and

$$v^\circ(0, x_0, \eta_0) = \min_\xi J(x_0, \eta_0; T)$$

In the following $\xi^\circ$ and $v^\circ$ are called optimal functions.

*Proof*. We shall consider the random trajectory $\{x_1(x_0, \eta_0; \eta, t)\}$ generated by the initial conditions $t_0 = 0$, $x_0$, $\eta_0$.

We shall average [2] the expression (2.1) with respect to the random quantities $x_1(x_0, \eta_0, t)$ and $\eta(t, \eta_0)$ and shall integrate on the interval of time $[0, T]$ the obtained equality

$$\frac{dM\{v(t, x, \eta)\diagdown x_0, \eta_0\}}{dt} = -M\{\omega(t, x, \xi)\diagdown x_0, \eta_0\} \tag{2.4}$$

$$v(0, x_0, \eta_0) = M\left\{\int_0^T \omega(t, x, \xi)\,dt + v(T, x(T), \eta(T))\right\} \tag{2.5}$$

Taking (1.5), (2.3) and (2.5) into consideration, we have

$$J(x_0, \eta_0; T) = v(0, x_0, \eta_0) \tag{2.6}$$

On the interval (0, $T$) the function $M\{v(t, x, \eta)\}$ is decreasing by virtue of (2.4) and as a consequence of the fulfillment of conditions (a) and (c), is positive in sign. We shall prove, that for $v = v^0$, $\xi = \xi^0$ the minimum of the integral (1.5) is attained.

Reasoning from the contrary, we shall assume the existence of a controlling quantity $\xi^*(t, x, \eta) \neq \xi^0(t, x, \eta)$ such that for the solutions $\{t, x(t), \eta(t), \xi^*(t)\}$ of equation (1.1) for $\xi = \xi^*$ and the initial conditions $x_0$ and $\eta_0$ the inequality

$$J_{\xi^*}(x_0, \eta_0; T) < J_{\xi^0}(x_0, \eta_0; T) \tag{2.7}$$

is satisfied.

From conditions (2.2) there follows

$$\left(\frac{dM\{v\}}{dt}\right)_{\xi^*} \geqslant -\omega(t, x, \xi^*) \tag{2.8}$$

Averaging and integrating (2.8) according to the stated scheme, we obtain

$$v(0, x_0, \eta_0) \leqslant M\left\{\int_0^T \omega(t, x, \xi^x)\,dt + v(T, x(T), \eta(T))\right\} = J_{\xi^*}(x_0, \eta_0; T) \tag{2.9}$$

The inequality (2.9) contradicts the equality (2.6) and the assumption made in (2.7). This proves the optimality of $\xi^0$ and $v^0$.

*Note 2.1.* It is assumed that the functions $v$, $\xi^0$ and $\xi^*$ are sufficiently smooth, so that the previously mentioned derivations are valid.

*Note 2.2.* The exposed statement of the problem and the approach to its solution are not changed if in the constitution of the right-hand side of the equation (1.1) there is an impulse disturbance $\gamma = \{\gamma_i\}$, the description of which is given in [2]. Nor does the presence of disturbances introduce any major differences in further considerations. Therefore the results of the present work can be generalized for systems which are under the influence of a disturbance of the indicated form (Section 8).

**3.** To establish the equations determining $v^0$ and $\xi^0$, it is necessary to know the expression of the derivative $dM\{v\}/dt$, formed by virtue of (1.1).

If the process $\eta(t)$ is discontinuous, then, assuming $\eta_1 \leqslant \eta \leqslant \eta_2$ we have [2]

$$\frac{dM\{v\}}{dt}=\frac{\partial v}{\partial t}+\sum_{n}\frac{\partial v}{\partial x_i}f_i(t,x,\eta,\xi)+$$
$$+\int_{\eta_1}^{\eta_2}v(t,x,\beta)\,d_\beta q(t,\eta,\beta)-q(t,\eta)\,v(t,x,\eta) \qquad (3.1)$$

We shall indicate the derivation of the expression $dM\{v\}/dt$ for a continuous process $\eta(t)$, by starting from considerations*, analogous to those made in [2,7] for the derivation of (3.1).

Let us assume that the function $v(t, x, \eta)$ is differentiable with respect to $t$, $x$, $\eta$ as many times as necessary. We determine the increment of the function $v$, using Taylor's expansion

$$\Delta v=\frac{\partial v}{\partial t}\Delta t+\sum_{n}\frac{\partial v}{\partial x_i}\Delta x_i+\frac{\partial v}{\partial \eta}\Delta\eta+\frac{1}{2}\frac{\partial^2 v}{\partial\eta^2}\Delta\eta^2+\ldots \qquad (3.2)$$

The discarded terms give after averaging a component of the order of magnitude $o(\Delta t)$. Computing the mathematical expectation of (3.2) for the initial conditions $\{x_i\}$, $t$, $\eta$, dividing by $\Delta t$ and taking the limit when $\Delta t \to 0$, we obtain, taking (1.3), (1.4) into consideration, the sought for formula

$$\frac{dM\{v\}}{dt}=\frac{\partial v}{\partial t}+\sum_{n}\frac{\partial v}{\partial x_i}f_i(t,x,\eta,\xi)+A_1(t,\eta)\frac{\partial v}{\partial\eta}+\frac{1}{2}B_1(t,\eta)\frac{\partial^2 v}{\partial\eta^2} \qquad (3.3)$$

The equations which determine the optimal functions $v^\circ(t, x, \eta)$, $\xi^\circ(t, x, \eta)$ in the continuous case are obtained** from conditions (2.1), (2.2) and by taking (3.3) into consideration

$$\frac{\partial v}{\partial t}+\sum_{n}\frac{\partial v}{\partial x_i}f_i(t,x,\eta,\xi)+A_1(t,\eta)\frac{\partial v}{\partial\eta}+\frac{1}{2}B_1(t,\eta)\frac{\partial^2 v}{\partial\eta^2}+\omega(t,x,\xi)=0 \qquad (3.4)$$

$$\sum_{n}\frac{\partial v}{\partial x_i}\frac{\partial f_i}{\partial\xi}+\frac{\partial\omega}{\partial\xi}=0 \qquad (3.5)$$

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* In terms of the theory of probable processes the derivative $dM\{v\}/dt$ is determined by an infinitely small linear generating operator [6]. A graphical explanation of the determination of the derivative is presented in [2].

** If the controlling quantity is restricted by an additional limitation, then this limitation must be taken into consideration when searching for the minimum of the left-hand side of (3.4).

Since it is difficult to solve equations (3.4) and (3.5) in the general case, it is possible to make an approximate construction of the functions $v^0$ and $\xi^0$ by using the method of introduction of a parameter in the right-hand sides of equation (1.1) and in the criterion of quality [2].

*Note 3.1.* In the discontinuous case, equation (3.4) is replaced by

$$\frac{\partial v}{\partial t} + \sum_n \frac{\partial v}{\partial x_i} f_i(t, x, \eta, \xi) + \int_{\eta_1}^{\eta_2} v(t, x, \beta)\, d_\beta q(t, \eta, \beta) - \\ - v(t, x, \eta)\, q(t, \eta) + \omega(t, x, \xi) = 0$$

4. Let us assume, that the transient response is described linearly by

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t, \eta) x_j + c_i(t, \eta)\xi \tag{4.1}$$

The quality of the process is evaluated by the integral

$$J(x_0, \eta_0;\ T) = M\left\{\int_0^T \left(\sum_{i,j}^{n} e_{ij} x_i x_j + \xi^2\right) dt + \sum_{i,j}^{n} d_{ij} x_i(T) x_j(T)\right\} \tag{4.2}$$

Here $e_{ij}$, $d_{ij}$ = const are the known coefficients of the positive definite forms $\omega$ and $r$. We shall search for an optimal function $v(t,x,\eta)$ of quadratic form

$$v(t, x, \eta) = \sum_{i,j}^{n} b_{ij}(t, \eta) x_i x_j \tag{4.3}$$

requiring furthermore, in agreement with the item (c) of Section 2

$$b_{ij}(T, \eta) = d_{ij} \tag{4.4}$$

If (4.1), (4.2), (4.3) are taken into account in the case of a continuous process $\eta(t)$, the system of equations (3.4), (3.5) takes the form

$$\sum_{i,j}^{n} \left[\frac{\partial b_{ij}}{\partial t} + A_1(t, \eta)\frac{\partial b_{ij}}{\partial \eta} + \frac{1}{2} B_1(t, \eta)\frac{\partial^2 b_{ij}}{\partial \eta^2}\right] x_i x_j + \\ + 2\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij} x_j\right)\left(\sum_{j=1}^{n} a_{ij} x_j + c_i \xi\right) = -\left(\sum_{i,j}^{n} e_{ij} x_i x_j + \xi^2\right) \tag{4.5}$$

$$\xi = -\sum_{i=1}^{n}\left(c_i \sum_{j=1}^{n} b_{ij}x_j\right) \tag{4.6}$$

Eliminating the function $\xi$ from (4.5), (4.6), we get, comparing the coefficients of the products $x_i x_j$

$$\frac{\partial b_{ks}}{\partial t} + A_1(t, \eta)\frac{\partial b_{ks}}{\partial \eta} + \frac{1}{2}B_1(t, \eta)\frac{\partial^2 b_{ks}}{\partial \eta^2} + \sum_{i=1}^{n}(b_{ki}a_{is} + b_{si}a_{ik}) - \\ -\left(\sum_{i=1}^{n} c_i b_{ki}\right)\left(\sum_{i=1}^{n} c_i b_{si}\right) = \begin{cases} -e_{ks} & \text{for } k = s \\ 0 & \text{for } k \neq s \end{cases} \tag{4.7}$$

The obtained system (4.7) consists of partial differential equations of the parabolic type. In order to solve it, in addition to the initial conditions (4.4), it is necessary, generally speaking, to assign also on the interval $\eta_1 < \eta < \eta_2$, the boundary conditions for $\eta = \eta_1$ and $\eta = \eta_2$ for all $t \in [0, T]$, determined by the assumed law of probability distribution $F(t, \alpha; \tau, \eta)$. Similarly, we assume (Section 1), that the realizations of $\eta^p(t)$ do not go outside the limits of the interval $(\eta_1, \eta_2)$ during the interval of time $[0, T]$ and attain inside this interval the points $t = T$. Then the conditions (4.4) define completely the solutions $b_{ks}(t, \eta)$ of the system (4.7).

*Note 4.1.* **A satisfactory physical example of a one-dimensional problem is given by the heat flow in a homogeneous bar of finite length, insulated on every side including the ends.**

For the discontinuous process $\eta(t)$, taking (3.1) into consideration, we have

$$\frac{\partial b_{ks}}{\partial t} + \sum_{i=1}^{n}(b_{ki}a_{is} + b_{si}a_{ik}) + \int_{\eta_1}^{\eta_2}[b_{ks}(t, \beta) - b_{ks}(t, \eta)]\, d_\beta q(t, \eta, \beta) - \\ -\left(\sum_{i=1}^{n} c_i b_{ki}\right)\left(\sum_{i=1}^{n} c_i b_{si}\right) = \begin{cases} -e_{ks} & \text{for } k = s \\ 0 & \text{for } k \neq s \end{cases} \tag{4.8}$$

The solution of the system of differential equations (4.8) is defined, once the initial conditions (4.4) are given.

**5.** Let us consider the question of the existence of a solution for the system (4.8). In the problem examined, any continuous controlling quantity $\xi^*$ which satisfies the condition of Lipschitz with respect to $x$ is admissible if the functional (1.5) is finite on the interval $[0, T]$ for $\xi = \xi^*$. Therefore, assuming that the admissible controlling quantities $\xi^*$ exist, we shall show, that among them, the solution (4.8) determines in a single manner the optimal $\xi^{0}$ securing the minimum of (4.2). Let us introduce the $m$-dimensional vector function

$$z(\eta) = \{z_i(\eta)\} = \{b_{11}(\eta), \ldots, b_{1n}(\eta), \ldots, b_{nn}(\eta)\} \qquad \left(\eta_1 \leqslant \eta \leqslant \eta_2,\ m = \frac{n(n+1)}{2}\right)$$

Let us represent by $\alpha = z(\eta)$ the elements of the space of the continuous vector functions $z(\eta)$ on the interval $[\eta_1, \eta_2]$. We shall determine the norm of the space $\{\alpha\}$ by the equality

$$\|\alpha\| = \sup_\eta |z(\eta)| = \sup_\eta \left| \left( \sum_m z_i^2 \right)^{\frac{1}{2}} \right|$$

The vector equality

$$\frac{\partial z(t, \eta)}{\partial t} = \varphi(t, z, \eta) \tag{5.1}$$

corresponding to the system (4.8), can be rewritten into the form

$$\frac{d\alpha}{dt} = f(t, \alpha) \tag{5.2}$$

Let us notice, that at every instant of time the right-hand side of (5.2) is an operator determined on the continuous functions $z(\eta)$, and, consequently, when changing to (5.2) we shall have an operator equation in the space $\{\alpha\}$.

We shall show that if the function $q(t, \eta, \beta)$ satisfies certain conditions, the operator $f(t, \alpha)$ actually transforms the elements of the space $\{\alpha\}$ into the space $\{\alpha\}$, i.e. transforms the continuous functions $\alpha = z(\eta)$ into continuous functions $\psi(\eta) = f[t, z(\eta)]$ for each $t \in [0, T]$.

Let us assume that the function $q(t, \eta, \beta)$ admits a density

$$q(t, \eta, \beta) = \int_{\eta_1}^{\beta} p(t, \eta, \gamma)\, d\gamma$$

whereupon $p(t, \eta, \beta)$ is a continuous function of its arguments.

On the right-hand side of (4.8) the continuity of all terms with respect to $\eta$ is obvious excepted, perhaps, the expression of the form

$$R(t, \eta) = \int_{\eta_1}^{\eta_2} [z(\beta) - z(\eta)]\, p(t, \eta, \beta)\, d\beta$$

for every $z(\eta)$. Let us consider an increment of $R(t, \eta)$ with respect to $\eta$

$$\Delta R = \int_{\eta_1}^{\eta_2} z(\beta)\, [p(t, \eta, \beta) - p(t, \eta', \beta)]\, d\beta - [z(\eta)\, q(t, \eta) - z(\eta')\, q(t, \eta')]$$

The function $p(t, \eta, \beta)$ determined on the closed interval $[\eta_1, \eta_2]$, is also continuous with respect to $\eta$. Thus, taking into consideration the equality

$$q(t, \eta) = \int_{\eta_1}^{\eta_2} p(t, \eta, \beta)\, d\beta$$

we get

$$\lim \Delta R = 0 \quad \text{for } \eta' \to \eta \tag{5.3}$$

which shows the continuity with respect to $\eta$ of the right-hand side of (4.8). It can also be shown that the operator $f(t, \alpha)$ in equation (5.2) is continuous with respect to $t$ and $\alpha$, and satisfies locally the condition of Cauchy-Lipschitz with respect to $\alpha$

$$\|f(t, \alpha') - f(t, \alpha'')\| \leqslant L\|\alpha' - \alpha''\| \qquad \text{for } t \in [0, T] \tag{5.4}$$

Then in the neighborhood of some initial point, the requirements for the local theorem of existence [8] of the solution of (5.2) are satisfied.

*Note 5.1.* In the more general case of the description of the process $\eta(t)$ by the functions $q(t, \eta)$, $q(t, \eta, \beta)$ the fulfillment of condition (5.3) and the continuity of the right-hand side of (5.2) with respect to $t$ are guaranteed if the function $q(t, \eta, \beta)$ bounded for $\eta \in [\eta_1, \eta_2]$ and non-decreasing with respect to $\beta$ is basically continuous [3, p.237] with respect to $\eta$ and $t$ (as well as with respect to the parameter). Then we can use for the proof the theorem of Kelly according to which the following equality takes place for the continuous function $z(t, \beta)$

$$\lim \int_{\gamma_1}^{\gamma_2} z(t', \beta)\, d_\beta q(t', \eta', \beta) = \int_{\gamma_1}^{\gamma_2} z(t, \beta)\, d_\beta q(t, \eta, \beta) \quad \text{for } t' \to t, \quad \eta' \to \eta$$

where $\gamma_1 < \beta < \gamma_2$ represent the points of continuity in the interval of variation of $\eta$.

Thus, if the function $q(t, \eta, \beta)$ is basically continuous with respect to $\eta$ and $t$ (in particular, when $q(t, \eta, \beta)$ has a continuous density $p(t, \eta, \beta)$), the equation (5.2) can be considered as an operator equation, whereupon its only solution exists in the neighborhood of the initial point.

The solution found in such a way, for the neighborhood of the point $\{T, \alpha\}$ (condition (4.4)) can be extended to the complete segment $[0, T]$ in the direction of decreasing $t$'s. The proof of such a possibility is analogous to the reasoning made in [9].

6. The approximate method described in [2] appears as a practical procedure for constructing the functions $v^{\circ}(t, x, \eta)$ and $\xi^{\circ}(t, x, \eta)$ without requiring solution of a system of equations of the type of (3.4) (3.5). We shall expose this method on the example of the linear system (4.1) when the process $\eta(t)$ is discontinuous.

Let us consider the auxiliary problem in which the transfer probability $\eta(t) = \alpha \to \eta(t) \leqslant \beta(\tau > t)$ is described in the following manner

$$P_{\vartheta}\{\eta(\tau) \leqslant \beta \setminus \eta(t) = \alpha\} = F(t, \alpha; \tau, \beta; \vartheta)$$

where $P_{\vartheta}$ is the conditional probability for some fixed value of the parameter $\vartheta$, which varies between the limits $0 \leqslant \vartheta \leqslant 1$ and is determined in such a manner that in the considered case the values of $\eta(t)$ satisfy the equality

$$q(t, \alpha; \vartheta) = \vartheta q(t, \alpha), \qquad q(t, \alpha, \beta; \vartheta) = \vartheta q(t, \alpha, \beta)$$

We shall search for an optimal function $v^{\circ}(t, x, \eta, \vartheta)$ of the form

$$v^{\circ} = \sum_{i,j}^{n} b_{ij}(t, \eta; \vartheta) x_i x_j$$

For $\vartheta = 0$, the problem for each $\eta \in (\eta_1, \eta_2)$ is considered as if it were determined for fixed values of the parameter $\eta = \eta_{\varphi}$. Taking (4.8) into consideration, we get

$$\frac{\partial b_{ks}}{\partial t} + \sum_{i=1}^{n} (b_{ki} a_{is} + b_{si} a_{ik}) - \left(\sum_{j=1}^{n} c_j b_{kj}\right)\left(\sum_{i=1}^{n} c_i b_{si}\right) = -\delta_{ks} e_{ks} \quad (k, s = 1, \ldots, n) \tag{6.1}$$

Taking the solution of (6.1) for some initial value, and varying afterwards the value of the parameter $\vartheta$ between 0 and 1, it is possible to find the solution of the initial problem, if the law of variation of $b_{ks}$ as a function of $\vartheta$ is known.

The function $v^{\circ}(t, x, \eta; \vartheta)$ is determined from the equation

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \left[\frac{\partial v}{\partial x_i} \sum_{j=1}^{n} a_{ij}(t, \eta) x_j\right] - \frac{1}{4}\left(\sum_{i=1}^{n} c_i \frac{\partial v}{\partial x_i}\right)^2 +$$

$$+ \vartheta \int_{\eta_1}^{\eta_2} [v(t, x, \beta; \vartheta) - v(t, x, \eta; \vartheta)]\, d_{\beta} q(t, \eta, \beta) = -\sum_{i,j}^{n} e_{ij} x_i x_j \tag{6.2}$$

Differentiating (6.2) with respect to $\vartheta$ and denoting for simplicity $\partial v / \partial \vartheta = g$, we obtain

$$\frac{\partial g}{\partial t}+\sum_{i=1}^{n}\left[\frac{\partial g}{\partial x_i}\sum_{j=1}^{n}a_{ij}(t,\ \eta)\,x_i\right]-\frac{1}{2}\left(\sum_{i=1}^{n}c_i\frac{\partial v}{\partial x_i}\right)\left(\sum_{i=1}^{n}c_i\frac{\partial g}{\partial x_i}\right)+$$
$$+\,\vartheta\int_{\eta_1}^{\eta_2}[g(t,\ x,\ \beta;\ \vartheta)-g(t,\ x,\ \eta;\ \vartheta)]\,d_\beta q(t,\ \eta,\ \beta)=$$
$$=-\int_{\eta_1}^{\eta_2}[v(t,\ x,\ \beta;\ \vartheta)-v(t,\ x,\ \eta;\ \vartheta)]\,d_\beta q(t,\ \eta,\ \beta) \qquad (6.3)$$

Substituting into (6.3) the form $v^{\circ}$, and comparing the coefficients of the terms in $x_i x_j$, we obtain a system of equations determining the solution $b_{ks}(t,\ \eta;\ \vartheta)$ for each value of $\vartheta$. Thus the solution found above for $\vartheta = 0$ can be extended for every $\vartheta$ up to unity*.

7. Let us show that if the functions $\xi^{\circ}$, $v^{\circ}$, are solutions of the given problem for the finite interval of time $[0,\ T]$, i.e. $v^{\circ}(t,\ x,\ \eta) = \min J(T)$ for $\xi = \xi^{\circ}$, then the optimal solution for $T = \infty$ can be found by taking the limit

$$v_\infty = \lim v^{\circ}(t,\ x,\ \eta), \qquad \xi_\infty = \lim \xi^{\circ}(t,\ x,\ \eta) \quad \text{for } T \to \infty$$

Let us look at the problem of finding

$$\min\nolimits_\xi J(x_0,\ \eta_0;\ \infty) = M\left\{\int_0^\infty \omega(t,\ x,\ \xi)\,dt\right\}$$

and let us compare it to the problem of the minimum of $J(x_0,\ \eta_0;\ T)$ investigated earlier, assuming $r[x] = 0$. Let us assume that the random parameter $\eta$ can only take a finite number of values $\{\eta_1,\ \ldots,\ \eta_k\}$ and that the transfer probability $\eta_i \to \eta_j$ is described by the matrix $\|p_{ij}\|_1^k$ in the following manner:

$$P\{\eta(t+\Delta t)=\eta_j \setminus \eta(t)=\eta_i\} = p_{ij}\Delta t + o(\Delta t)$$

For the problem considered in Sections 4 and 5 for $T = \infty$, the controlling function which, in addition to the finiteness of the functional (4.2) insures also the asymptotic stability with respect to the probability of the solution $x = 0$ of the system (4.1) is admissible. This

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* The proof of the possibility of such an extension in principle is not considered here. As was noted in [2], this question is closely related to the question of the existence of a solution for the system (4.8). It can be confirmed that such a possibility follows from the existence of equation (4.8).

is equivalent to the existence requirement of a positive definite quadratic form $v(t, x, \eta)$ such that for the admissible $\xi$, we have*

$$\frac{dM\{v\}}{dt} \leqslant -k(x_1^2 + \ldots + x_n^2) \qquad (k > 0)$$

Using this criterion, it is possible to indicate a sufficient condition of existence of an admissible controlling quantity for the linear system (4.1).

*Theorem 7.1.* Let at any fixed instant of time, for a constant value of $\eta$ of the interval $[\eta_1, \eta_2]$, the following conditions be fulfilled:

1. The system of vectors $c(t, \eta)$, $A(t, \eta)c(t, \eta)$, ..., $A^{n-1}(t,\eta)c(t,\eta)$ is linearly independent.

2. The following bounds exist

$$\left|\frac{\partial a_{ij}(t, \eta)}{\partial \eta}\right| \leqslant N, \qquad \left|\frac{\partial c_i(t, \eta)}{\partial \eta}\right| \leqslant N \tag{7.1}$$

It is then possible to find numbers $D > 0$ and $\kappa > 0$ such that if the inequalities

$$\left|\frac{\partial a_{ij}}{\partial t}\right| \leqslant D, \qquad \left|\frac{\partial c_i}{\partial t}\right| \leqslant D \qquad \sum_{l \neq s}^{k} p_{sl} |\eta_l - \eta_s| \leqslant \varkappa \tag{7.2}$$

are satisfied, it is possible to design an admissible controlling quantity.

The proof of Theorem 7.1 is carried out accordingly to the model given in [10, p.832], taking into consideration in the present case the non-stationary behavior of the system (4.1).

*Note 7.1.* It is possible to prove a statement analogous to Theorem 7.1 for a more general description of the discontinuous process $\eta(t)$ by

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* The solution $x = 0$ will be called asymptotically stable with respect to the probability, if for any two numbers $\varepsilon > 0$ and $p > 0$, it is possible to find a $\delta > 0$ such that the inequality

$$P\{(x_1^2 + \ldots + x_n^2) < \varepsilon^2 \setminus (x_{10}^2 + \ldots + x_{n0}^2) \leqslant \delta^2\} > 1 - p$$

is satisfied, and furthermore, for any number $\omega > 0$

$$\lim_{t \to \infty} P\{(x_1^2(t) + \ldots + x_n^2(t)) < \omega\} = 1$$

the functions $q(t, \alpha)$, $q(t, \alpha, \beta)$, and also for a continuous process $\eta(t)$. In the first case, the last of the inequalities (7.2) in the conditions of the theorem is replaced by

$$\left(\frac{dM\{|\eta(t)-\alpha|\}}{dt}\right)_{\tau,\alpha} = \left(\frac{dM\{|\eta(t)-\alpha \diagdown |\eta(\tau)=\alpha\}}{dt}\right)_{t=\tau} \leqslant \varkappa_1 \qquad (\varkappa_1>0) \tag{7.3}$$

and in the second case, taking (1.3), (1.4) into consideration

$$|A_1(\tau,\alpha)| = \left|\frac{dM\{\eta(t)-\alpha\}}{dt}\right|_{\tau,\alpha} \leqslant \varkappa_2 \quad (\varkappa_2>0) \tag{7.4}$$
$$B_1(\tau,\alpha) = \left(\frac{dM\{[\eta(t)-\alpha]^2\}}{dt}\right)_{\tau,\alpha} \leqslant \varkappa_2$$

*Note 7.2.* The Theorem 7.1 remains valid if the bounds (7.2), (7.3) or (7.4) are given in the average on a certain interval of time $[t_0, t_0+\tau]$, i.e. if the following conditions are satisfied

$$\left|\frac{\partial a_{ij}}{\partial t}\right| \leqslant \Phi_1(t), \qquad \left|\frac{\partial c_i}{\partial t}\right| \leqslant \Phi_1(t) \qquad \sum_{l\neq s}^{k} p_{sl}\,|\eta_l-\eta_s| \leqslant \Phi_2(t) \tag{7.5}$$

where $\Phi_1(t)$, $\Phi_2(t)$ are some continuous functions subject to the inequalities

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau} \Phi_1(t)\,dt \leqslant \varkappa_3, \qquad \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \Phi_2(t)\,dt \leqslant \varkappa_4$$

($\kappa_3$, $\kappa_4 > 0$ are sufficiently small constants).

Let us pursue the examination of the linear system (4.1). Let us consider as admissible the controlling quantities existing for $0 < T \leqslant \infty$, and consequently

$$\lim_{T\to\infty}\int_{\tau}^{T} M\{\omega(t, x, \xi^*)\}\,dt < \infty \tag{7.6}$$

for some admissible $\xi = \xi^*$.

But from (7.6) there follows that the function $v^0(\tau, x, \eta)$ for any fixed $\tau = t$, $x = x(t)$, $\eta = \eta(t)$ is bounded for all $T \geqslant \tau$. Since in such a case $v^0$ does not decrease when $T$ increases, there is a limit $v_\infty = \lim v^0$, whereupon the function $v_\infty(\tau, x, \eta)$ is positive definite. It remains to show that the function $v_\infty$ and the corresponding controlling quantity $\xi_\infty$ are solutions of the optimal problem when $T = \infty$.

Let us consider in detail the stationary case, i.e.

$$\omega = \omega(x, \xi), \qquad \frac{dx}{dt} = f(x, \eta, \xi), \qquad \xi = \xi(x, \eta)$$

Then for $T = \infty$, any instant of time can be taken as an initial point. Thus it is sufficient to investigate for the case of the function $v^{0}(0, x, \eta)$. We have

$$v^{\circ}(0, x, \eta) = M\left\{\int_{0}^{T} \omega(x, \xi^{\circ})\, dt\right\} = \min_{\xi} M\left\{\int_{0}^{T} \omega(x, \xi)\, dt\right\} \tag{7.7}$$

For $T = \infty$, the functional (7.7) has a lower limit $v^{*}$ with respect to the set of admissible $\xi = \xi^{*}$. Let us assume the contrary, i.e.

$$v^{*} = \inf_{\xi} M\left\{\int_{0}^{\infty} \omega(x, \xi)\, dt\right\} \neq v_{\infty}$$

Let, for instance

$$v^{*} < v_{\infty} \tag{7.8}$$

But then, there is a $\xi_{\infty}^{**}$ such that the inequality

$$v^{**}(0, x, \eta) = M\left\{\int_{0}^{\infty} \omega(x, \xi^{**})\, dt\right\} < v_{\infty}$$

is satisfied.

It is obvious that $v^{0} < v^{**}$. When $T$ increases the function $v^{0}$ converges towards a value equal to $v_{\infty}$, therefore we get at the limit, when $T = \infty$

$$v_{\infty} \leqslant v^{**}$$

which contradicts the assumption (7.8). Let now

$$v_{\infty} < v^{*} \tag{7.9}$$

We shall denote by $\xi_{0}^{*}(0, x, \eta) = \lim \xi^{0}(0, x, \eta)$ when $T \to \infty$. The existence of that limit and the linearity of $\xi_{0}^{*}(0, x, \eta)$ with respect to $x$ follows from the existence of the limit of $v^{0}(0, x, \eta)$ when $T \to \infty$ and from the formula (4.6). We shall note the uniformity of $\xi_{0}^{*}$ with respect to $\eta$ when $x$ is fixed.

Since $v^{*}$ is the lower limit with respect to $\xi$ of the optimized functional, the inequality (7.9) cannot be satisfied if $\xi_{0}^{*}$ belong to the admissible controlling values and furthermore, if

$$M\left\{\int_{0}^{\infty} \omega(x, \xi_{0}^{*})\, dt\right\} = v_{\infty} \tag{7.10}$$

We shall show, that in fact, (7.10) is fulfilled. Let us choose a number $T_{1} > T$ such that at each instant of time of the interval $0 \leqslant t \leqslant T$

the following inequality is fulfilled (for all possible $\eta$)

$$|\xi_1(0, x, \eta) - \xi_0^*(0, x, \eta)| \leqslant \delta \tag{7.11}$$

where $\delta$ is a number chosen *a priori* as small as desired, and $\xi_1$ corresponds to the interval $[0, T_1]$.

As a consequence of the integral continuity there follows from (7.11) that

$$\left| M\left\{\int_0^T \omega(x, \xi_1)\,dt\right\} - M\left\{\int_0^T \omega(x, \xi_0^*)\,dt\right\} \right| \leqslant \gamma(\delta)$$

whereupon $\gamma \to 0$ when $\delta \to 0$.

Hence

$$M\left\{\int_0^T \omega(x, \xi_0^*)\,dt\right\} < M\left\{\int_0^T \omega(x, \xi_1)\,dt\right\} + \gamma(\delta) \leqslant v_\infty + \gamma(\delta) \tag{7.12}$$

By virtue of the arbitrary smallness of $\gamma$, there results from (7.12) that for all $T > 0$

$$M\left\{\int_0^T \omega(x, \xi_0^*)\,dt\right\} \leqslant v_\infty \tag{7.13}$$

From (7.13) it follows that for $T = \infty$ we obtain (7.10). This completes the proof. Thus

$$v_\infty(0, x, \eta) = \min\nolimits_\xi M\left\{\int_0^\infty \omega(x, \xi)\,dt\right\} = \lim_{T\to\infty} v^\circ(0, x, \eta)$$

$$\xi_\infty(0, x, \eta) = \lim_{T\to\infty} \xi^\circ(0, x, \eta)$$

Therefore, from the existence proved in Section 5 of a solution in the interval $[0, T]$ for linear systems and the existence of an admissible controlling quantity when $T = \infty$, there follows the existence of a solution of the optimal problem when $T = \infty$.

The considered limit process can be used for approximate calculations when $T = \infty$.

**8.** Let us present the solution of the problem when the transient response is described by the equation

$$\frac{dx}{dt} = A(t, \eta)x + c(t, \eta)\xi + \gamma(t) \tag{8.1}$$

where the vector function $\{\gamma_i(t)\}$ is a random impulse disturbance.

It is assumed that $\gamma_i(t)$ can be represented in the form

$$\gamma_i = \sum_k \mu_i \nu_i(t_k)\,\delta(t - t_k)$$

Here $\nu_i(t)$ is a random value, the vector $\mu = \{\mu_i\}$ is a constant and $\delta(t)$ represents the $\delta$-function.

The random moments $t_k$ of occurrence of the impulses are distributed on the $t$-axis according to a Poisson distribution with frequency $\lambda$. It is possible to compute [7] the size of the unit step of the $i$th coordinate $\Delta x_i \approx \mu_i \nu_i$ originating during a sufficiently small interval of time $\Delta t$ with a probability $P[\Delta t] \approx \lambda\,\Delta t$. Let us assume that $M\{\nu_i\} = 0$; the dispersion matrix $\|\sigma_{ij}\|_1^n$ which has for coefficients $\sigma_{ij} = M\{\nu_i \nu_j\}$ is known. (A detailed description of such disturbance if presented in [11, p.63].)

In other respects the equation (8.1) is not different from (4.1). We shall evaluate the quality of the transient response on the interval $[0, T]$ with the integral (4.2).

For equation (4.1) when $\{\gamma_i\} \equiv 0$, the optimal function $v(t, x, \eta)$ on the interval $[0, T]$ can be found (Section 4) in the form of the quadratic form (4.3) from equations of the form (4.5) to (4.6) (or (4.6), (4.8)) for the given initial conditions (4.4).

Therefore the condition (2.1) takes the form

$$\left(\frac{dM\{v\}}{dt}\right)_{\gamma=0} = -\sum_{i,j}^{n} e_{ij}x_i x_j - \xi^2 \tag{8.2}$$

Calculating the expression of the derivative $dM\{v\}/dt$, we get [10] for $\gamma \neq 0$ by virtue of system (8.1)

$$\frac{dM\{v\}}{dt} = \left(\frac{dM\{v\}}{dt}\right)_{\gamma=0} + s(t, \eta) \qquad \left(s(t, \eta) = \lambda \sum_{i,j}^{n} b_{ij}(t, \eta)\,\mu_i \mu_j \sigma_{ij}\right) \tag{8.3}$$

where $s(t, \eta)$ is a non-negative function.

We shall now construct a function $V(t, x, \eta)$ in the following manner:

$$V(t, x, \eta) = v(t, x, \eta) + v_1(x, \eta) \tag{8.4}$$

On the right-hand side of (8.4) the first component is the quadratic form found above, and the second is determined by the conditions

$$\frac{dM\{v_1\}}{dt} = -s(t, \eta), \qquad v_1(T, \eta(T)) = 0 \tag{8.5}$$

We notice that by using (8.5) it can be shown that

$$V(t, x, \eta) = \sum_{i,j}^{n} b_{ij}(t, \eta) x_i x_j + M\left\{\int_t^T s(\tau, \eta(\tau))\, d\tau\right\}$$

Taking (8.2) to (8.5) into consideration, we find for the derivative $dM\{V\}/dt$ by virtue of equation (8.1)

$$\frac{dM\{V\}}{dt} = -\sum_{i\ j}^{n} e_{ij} x_i x_j - \xi^2 \tag{8.6}$$

It can be verified that the constructed function $V(t, x, \eta)$ satisfies the optimum condition (2.2). Therefore, this function is optimal. The optimal controlling quantity $\xi^{\circ}$ corresponding to $V$, remains the same as in the case $\{\gamma_i\} \equiv 0$, i.e. $\xi^{\circ}$ can be found from formula (4.6).

*Note 8.1.* If $\{\gamma_i\} \neq 0$, then

$$M\left\{\sum_{i,j}^{n} e_{ij} x_i x_j\right\}$$

does not tend towards zero when $T \to \infty$. Therefore the integral (4.2) when $T \to \infty$ does not converge. We shall evaluate the quality of the process by the criterion

$$J^{(\gamma)}(x_0, \eta_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T M\left\{\sum_n x_i^2 + \xi^2\right\} dt \tag{8.7}$$

The solution of the problem under such conditions is accomplished by taking the limit, when $T \to \infty$, of an optimal solution $\{V, \xi^{\circ}\}$ of the interval $[0, T]$. The limiting process is similar to the one given above. As a result we shall find that

$$\xi_\infty = \lim_{T \to \infty} \xi^{\circ}, \qquad \min_\xi J^{(\gamma)}(x_0, \eta_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T M\{s(t, \eta(t))\}\, dt$$

As this work was being completed the author became acquainted with the work of Kalman [12], where randomly varying systems of the discrete type are considered, whereupon a limiting process analogous to the one described in Section 7, is used in the proofs.

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